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The Density of Translates of Zonal Kernels on Compact Homogeneous Spaces

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Compact manifolds embedded in Euclidean space which have a transitive group G of linear isometries, such as the spheres with the rotation group or the "flat" tori with the group of rotations in each coordinate direction, admit a natural notion of a continuous G-invariant kernel function $k(\mathbf{x}, \mathbf{y})$, which generalizes the idea of a radial or distance-dependent function on the spheres and tori. In connection with a study of quasi-interpolation on these spaces, we have reproved and extended results of Sun for the spheres to characterize those kernels for which the span of the translates, $\sum a_n k(\mathbf{x}, \mathbf{y}_n)$, is dense in the continuous functions. The essence of the characterization is that the integral operator with G-invariant kernel $k(\mathbf{x}, \mathbf{y})$ must be non-singular when restricted to the space of nth degree polynomial functions. This requires that the polynomials be invariant under all such linear operators, which is true for many compact homogeneous M including the spheres, tori, and others. In fact the non-singularity must hold only on any finite-dimensional space of zonal polynomials, those which are pointwise fixed by the subgroup of all isometries fixing a single point on M. In practical terms this later condition is verified by choosing one point on the manifold (the north pole on the spheres or the identity element on the flat tori), picking some basis for the polynomials of

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given degree which are fixed under the isometries leaving the pole invariant, and testing whether the integral operator (which leaves this space invariant) has a non-singular matrix. In all the cases considered, where the family of G-invariant kernels lead to commuting operator families, there are diagonalizing bases for this restricted operator, and the characterization becomes the non-vanishing of the appropriate Fourier-like coefficients.

1. INTRODUCTION

For any smooth compact submanifold M of $\mathbb{R}^d$ with $\langle \mathbf{x}, \mathbf{y} \rangle$ the standard inner product, the restrictions of the polynomials form a dense subspace of $\mathbf{C}(M)$ and so provide a natural class of functions for either direct approximation or analysis of the approximating properties of other classes of functions. In one particular case of interest, quasi-interpolation, the object is often the study of real symmetric (or self-adjoint) kernels $k(\mathbf{x}, \mathbf{y})$ on $M \times M$ and the approximation properties of the linear combinations $\sum a_i k(\mathbf{x}, \mathbf{y}_i)$. A first step in understanding such properties is to investigate the density properties of all such linear combinations or more generally:

$$\text{Characterize the closure of } \mathscr{K} = \left\{ \sum a_i k(\cdot, \mathbf{y}_i) \colon a_i \in \mathbb{R}, \mathbf{y}_i \in M \right\}.$$

(In case k is complex-valued and self-adjoint, $a_i \in \mathbb{R}$ is replaced by $a_i \in \mathbb{C}$.) How to use the well-understood density of the polynomials to determine $\bar{\mathscr{K}}$ for specific k forms the major focus for this paper.

For spheres or tori characterization of $\bar{\mathscr{K}}$ has been achieved previously by the use of facts about spherical harmonics or multiple Fourier series [3, 7, 8]. Our aim here is to simplify and unify these cases by dispensing with these facts and using only extremely elementary facts from geometry, analysis, and approximation theory. We can make substantial progress on the general characterization problem for those compact manifolds in a Euclidean space which are homogeneous and reflexive in the sense that their local geometry looks the same at any two points $\mathbf{x}$, $\mathbf{y}$, and an isometry which transfers the geometry from $\mathbf{x}$ to $\mathbf{y}$ can be found which also maps $\mathbf{y}$ back to $\mathbf{x}$. That is, we study *embedded compact reflexive spaces* in the sense of

Definition 1.1. A compact manifold $M \subset \mathbb{R}^d$ in an *embedded compact homogeneous space* provided

(i) There is a compact group $G \subseteq O(d)$ of orthogonal matrices which acts as a transitive group of isometries on M, i.e. $g \cdot M = M$ for all $g \in G$ and for each pair $\mathbf{x}, \mathbf{y} \in M$, there is an isometry $g \in G$ with $g \cdot \mathbf{x} = \mathbf{y}$.

M is an *embedded compact reflexive space*[2] provided it also satisfies

(ii) For each pair $\mathbf{x}, \mathbf{y} \in M$ there is a $g \in G$ with $g \cdot \mathbf{x} = \mathbf{y}$ and $g \cdot \mathbf{y} = \mathbf{x}$.

Remark 1.2. The definitions here are only for *compact* M, so we will often drop the word *compact*. Moreover, since the metric and algebraic properties of M we require are completely determined by its embedding in a specific Euclidean space, none of the abstract theory of homogeneous spaces will be needed; all the development will depend on relatively familiar facts such as the isometric action given by multiplication by an orthogonal matrix and the invariance of Lebesgue measure under an orthogonal change of variables.

Harmonic analysts have long studied functional analysis in this setting by means of group representation theory [2]. In fact much of the material in Sections 3 and 4 is familiar to them. From the approximation theorist's perspective the definition of *embedded reflexive spaces* captures the geometric properties of a space which render the familiar polynomials (of a fixed degree) invariant under both the linear isometries from G and the action of G-invariant kernels and thus guarantees that the polynomials provide an intrinsic link between the geometric structure and functional analysis on the manifold. This allows much of the complexity of representation theory and associated differential operators to be replaced by simple arguments about ordinary polynomials and finite dimensional invariant subspaces.

For example, for any sphere about the origin, the orthogonal matrices form a transitive group of linear isometries, and for any $\mathbf{x}$, $\mathbf{y}$, reflection through or rotation around the axis $(\mathbf{x}+\mathbf{y})/2$ provides the required interchange of $\mathbf{x}$, $\mathbf{y}$. Any orthogonal map composed with a polynomial is again a polynomial of the same degree. Here the polynomials are just finite series of spherical harmonics, i.e., eigenfunctions of the spherical Laplacian, but no facts about spherical harmonics or the Laplace operator on the sphere are needed for our work. Similarly the m-torus embedded in $\mathbb{R}^{2m}$, realized as a product of m circles about the origin in $\mathbb{R}^2$, possesses the torus itself as a transitive group of isometries realized by the block diagonal orthogonal matrices with 2×2 rotations on the diagonal. However, to have a reflexive space we must also include the reflections in each $\mathbb{R}^2$. Any polynomial in this setting is just a finite multivariate trigonometric series, but here, too, no facts about multi-dimensional Fourier series are needed. A less familiar example, but one which begins to get at the diversity of the

[2] This slight relaxation of the standard definition of compact symmetric space is one way to make the essence of the analysis a commutative theory, e.g. Proposition 3.7.

examples we could consider, is the space of 2-dimensional subspaces of $\mathbb{R}^n$ realized as the manifold of all rank 2 projection matrices in $\mathbb{R}^{n\times n}$. Since any projection has the form $\mathbf{v}_1\mathbf{v}_1^T+\mathbf{v}_2\mathbf{v}_2^T$, where $\{\mathbf{v}_1, \mathbf{v}_2\}$ is an orthonormal basis for the subspace, orthogonal maps of $\mathbb{R}^n$ act transitively on these bases and yield a transitive action on the rank 2 projection matrices. The interchange of two 2-planes, $\mathbf{x}$, $\mathbf{y}$, arises from the Euclidean reflection through a 2-plane which bisects the (angular) distance between $\mathbf{x}$ and $\mathbf{y}$.

For embedded homogeneous manifolds M we study *G-invariant kernels* and their associated *zonal functions*. The specific definitions are in

DEFINITION 1.3. Let M be an embedded homogeneous space and G a transitive group of linear isometries.

(i) A kernel function k on $M\times M$ is *G-invariant* if $k(\mathbf{x}, \mathbf{y}) = k(g\cdot\mathbf{x}, g\cdot\mathbf{y})$ for all $g\in G$.

(ii) A *zonal function* on M with pole $\mathbf{p}\in M$ is any function f with $f(h^{-1}\cdot\mathbf{x}) = f(\mathbf{x})$ for all $h\in G$ with $h\cdot\mathbf{p}=\mathbf{p}$ and all $\mathbf{x}\in M$.

Each G-invariant kernel, k, determines a zonal function with pole $\mathbf{p}$: $k_{\mathbf{p}}(\mathbf{x}) = k(\mathbf{x}, \mathbf{p})$ and conversely. Obviously the structure of a G-invariant kernel $k(\mathbf{x}, \mathbf{y})$ depends only on the relative geometry between its variables.

In the case of the spheres, the invariance of the kernels under all orthogonal maps means that the kernel, or rather the associated zonal function, only depends on the great circle distance between its arguments; thus we study radial functions on spheres which are essentially functions of one real variable. In the case of the tori, the invariance of a kernel under rotations means the associated zonal function is just an arbitrary continuous function on the torus. In the case of 2-dimensional subspaces in $\mathbb{R}^n$, the kernels depend only on two (dihedral) angles which characterize the relative placement of the two planes, i.e., a function of two circular distances.

When we require M to be *reflexive*, any G-invariant kernel will be *symmetric* since we can use the g from Definition 1.3(ii) to conclude $k(\mathbf{x}, \mathbf{y}) = k(g\cdot\mathbf{x}, g\cdot\mathbf{y}) = k(\mathbf{y}, \mathbf{x})$. At least for *real* k this suggests we may be dealing with some self-adjoint integral operators for which identification of the eigenfunctions may be an appropriate and useful goal.

We shall associate to any (continuous) kernel k on $M\times M$ the operator T_k defined by

$$T_k(f)(\mathbf{x}) = \int_M k(\mathbf{x}, \mathbf{y})\, f(\mathbf{y})\, d\mu(\mathbf{y}),$$

where μ is the "surface" measure on M. This measure is obtained by splitting the Lebesgue measure of the ambient space as a local product $dx = dx_M \times dx_{M^\perp}$ of its components tangential and normal to M. Then $d\mu = c_M\, dx_M$ is just the tangential component normalized by the constant c_M so $\mu(M) = 1$.

Now the existence of eigenfunctions for any single operator T_k is a well-known fact. However, in our study of G-invariant k the set

$$\mathcal{T}_G = \{T_k : k \text{ any } G\text{-invariant continuous kernel}\}$$

is often a commuting family closed under adjoints. For instance, this is so when M is reflexive (Proposition 3.7) or a flat torus. As a result there are finite-dimensional spaces of *polynomials* which are eigenfunctions for the whole family $\mathcal{T}_G$, and depend *only* on the linear isometries, i.e., geometric invariants, of M.

Our main goal is to prove the following result (see Theorem 4.3):

THEOREM 1.4. *Suppose M is an embedded compact homogeneous space with G an associated group of linear isometries of M. Let H_n be the polynomials of degree at most n which are orthogonal to those of degree less than n in $L^2(M)$. Fix a pole $\mathbf{p} \in M$ and let $K_\mathbf{p}$ be the isometries in G which fix $\mathbf{p}$. Let ${}^{K_\mathbf{p}}H_n$ be the zonal polynomials in H_n with pole $\mathbf{p}$. Suppose M is reflexive or more generally that $\mathcal{T}_G$ is a commuting family. If $k(\mathbf{x}, \mathbf{y})$ is a continuous G-invariant kernel on $M \times M$, then $\mathcal{K}$ is dense in $\mathbf{C}(M)$ if and only if for every n and all $p \in {}^{K_\mathbf{p}}H_n$, $T_k(p)(\cdot) \neq 0$. In particular for any basis $\{p_{n,j}\}$ of ${}^{K_\mathbf{p}}H_n$ consisting of eigenfunctions with $T_k(p_{n,j}) = a_{n,j}\, p_{n,j}$, density holds if and only if $a_{n,j} \neq 0$ for all n, j.*

Remark 1.5. Since the zonal polynomials are T_k invariant (see Theorem 3.8 below) and the set $\mathcal{T}_G$ is a commuting (normal) family there always exist eigenfunction bases which are independent of k. Thus this theorem leads to quite explicit tests for density of $\mathcal{K}$.

In Section 2 we consider the special case of S^{d-1}, the sphere in $\mathbb{R}^d$. The treatment of the sphere gives insight into the general procedure described in Section 3 and the following section. In Section 5 the sphere is revisited in light of the preceding work, and then the example of the tori is discussed.

2. THE SPHERE

In this section we illustrate the details of the general setting discussed above in the simple case of S^{d-1}, the sphere in $\mathbb{R}^d$. In Proposition 2.1 of this section we reformulate the general density result of Theorem 1.4 as a known result of Sun [7]. We use this opportunity to provide a simplified proof more in the spirit of our geometric invariant approach and out Theorem 4.1 in Section 4. Here, we study kernels $k(\mathbf{x}, \mathbf{y})$ which are invariant under all rotations and reflections, and thus depend only on the geodesic distance between $\mathbf{x}$ and $\mathbf{y}$. Since the geodesic distance, $d(\mathbf{x}, \mathbf{y}) = \cos^{-1}(\langle \mathbf{x}, \mathbf{y} \rangle)$, is a strictly monotone (decreasing) function of $\langle \mathbf{x}, \mathbf{y} \rangle$, we wish to approximate by functions of the form $k(\cdot, \mathbf{y}) = \phi(\langle \cdot, \mathbf{y} \rangle)$, where $\mathbf{y} \in S^{d-1}$. If we restrict the polynomials of degree n to S^{d-1} we obtain a space P_n. If we let $H_n = P_n \cap P_{n-1}^{\perp}$, then this polynomial space turns out to be exactly the *harmonic polynomials* of degree n. The classical spherical harmonics $h_{n1}, h_{n2}, ..., h_{n\pi_n}$ are an orthonormal basis for H_n with respect to the (normalized) surface measure Ω on S^{d-1}. Using the Sone–Weierstrass Theorem (see, e.g., Cheney [1]), we know that the space $H = \bigcup P_n = \bigcup \bigoplus_{j=0}^{j=n} H_j$ of all polynomials is uniformly dense in $\mathbf{C}(S^{d-1})$ and hence the collection of all spherical harmonics, $\{h_{ni}\}$, over all degrees, is an orthonormal basis for $L^2(S^{d-1})$. It is well known that the Gegenbauer polynomials p_n^{λ}, $\lambda = (d-2)/2$, satisfy the relationship

$$p_n^{\lambda}(\langle \mathbf{x}, \mathbf{y} \rangle) = \sum_{i=1}^{\pi_n} h_{ni}(\mathbf{x})\, h_{ni}(\mathbf{y}), \tag{1}$$

where the p_n^{λ} are normalized so that $p_n^{\lambda}(1) = \dim H_n = \pi_n$ as required by integration of 1 over the diagonal $\mathbf{x} = \mathbf{y}$; see Muller [5]. However, this relationship holds true for any orthonormal basis, not just the classical spherical harmonics. (In effect we give a simple (re)proof of this via Proposition 3.6 and the discussion of the sphere in Section 5.)

We can now prove

PROPOSITION 2.1. *Let* $\phi: [-1, 1] \to \mathbb{R}$ *be continuous and have the Gegenbauer expansion*

$$\phi(t) = \sum_{n=0}^{\infty} a_n p_n^{\lambda}(t).$$

Then, the set $\Phi = \{\phi(\langle \cdot, \mathbf{y} \rangle) : \mathbf{y} \in S^{d-1}\}$ *is fundamental in* $\mathbf{C}(S^{d-1})$ *if and only if* $a_n \neq 0$ *for any* $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Proof. Let us prove only the "if" direction first. Because the space H is dense in $\mathbf{C}(S^{d-1})$ we need only prove that we can approximate uniformly any h_{ni}, $n = 0, 1, 2, ..., i = 1, 2, ..., \pi_n$ by a function of the form

$$\sum_{i \in \mathcal{I}} c_i \phi(\langle \cdot, \mathbf{y}_i \rangle),$$

where $\mathscr{I}$ is some finite index set. Then, for any $n = 0, 1, 2, \ldots$ and $i = 1, 2, \ldots, \pi_n$,

$$\begin{aligned}
&\int_{S^{d-1}} \phi(\langle \mathbf{x}, \mathbf{y} \rangle)\, h_{ni}(\mathbf{y})\, d\Omega(\mathbf{y}) \\
&\quad = \int_{S^{d-1}} \left\{ \sum_{m=0}^{\infty} a_m p_m^{\lambda}(\langle \mathbf{x}, \mathbf{y} \rangle) \right\} h_{ni}(\mathbf{y})\, d\Omega(\mathbf{y}) \\
&\quad = \int_{S^{d-1}} \left\{ \sum_{m=0}^{\infty} a_m \sum_{j=1}^{\pi_m} h_{mj}(\mathbf{x})\, h_{mj}(\mathbf{y}) \right\} h_{ni}(\mathbf{y})\, d\Omega(\mathbf{y}) \\
&\quad = a_n h_{ni}(\mathbf{x}),
\end{aligned}$$

where the penultimate line follows from (1) and the final step uses the orthonormality of the spherical harmonics. This string of $L^2(S^{d-1})$ equalities has continuous functions at each end, so we get the pointwise equality

$$\int_{S^{d-1}} \phi(\langle \mathbf{x}, \mathbf{y} \rangle)\, h_{ni}(\mathbf{y})\, d\Omega(\mathbf{y}) = a_n h_{ni}(\mathbf{x}), \tag{2}$$

the Funk–Hecke formula [5]. The result follows by uniform approximation of the above integral by an appropriate Riemann sum, since the integrand is jointly uniformly continuous.

For the "only if" direction we show that if $a_n = 0$ for some n then $h_{nj} \notin \overline{\mathrm{Span}(\Phi)}$ for any $j = 1, 2, \ldots, \pi_n$. In fact the equality (2) above shows each $h_{nj}(\mathbf{x})$ is orthogonal to $\phi(\langle \mathbf{x}, \mathbf{y} \rangle) = \phi(\langle \mathbf{y}, \mathbf{x} \rangle)$ and hence to the span of Φ. Taking uniform limits preserves this orthogonality and proves the "only if" part. ∎

Remark 2.2. Let $\mathscr{N} = \{n : a_n = 0\}$ be finite. Then, to ensure a dense approximating subspace, Φ must be augmented by $H_{\mathscr{N}} = \mathrm{Span}(\bigcup_{n \in \mathscr{N}} H_n)$. Because of the orthogonality of the spherical harmonics the coefficients c_i in the approximation

$$f(\mathbf{x}) \approx f_{\mathscr{N}}(\mathbf{x}) + \sum_{i \in \mathscr{I}} c_i \phi(\langle \cdot, \mathbf{y}_i \rangle),$$

with $f_{\mathscr{N}} \in H_{\mathscr{N}}$, can be chosen so that

$$\sum_{i \in \mathscr{I}} c_i h_{nj}(\mathbf{y}_i) = 0,$$

for every $j = 0, 1, ..., \pi_n$ and $n \in \mathcal{N}$. For, suppose $m \notin \mathcal{N}$ and $h_{mj} \in H_m$. Then

$$\int_{S^{d-1}} h_{mj}(\mathbf{y})\, h_{ni}(\mathbf{y})\, d\Omega(y) = 0,$$

for all $h_{ni} \in H_{\mathcal{N}}$. If we choose a sufficiently accurate quadrature rule with nodes $\{\mathbf{y}_i\}$ and weights w_i which preserves this orthogonality, we have

$$\sum_{i \in \mathcal{I}} w_i h_{mj}(\mathbf{y}_i)\, h_{ni}(\mathbf{y}_i) = 0.$$

Setting $c_i = w_i h_{mj}(\mathbf{y}_i)$, we have

$$\begin{aligned} h_{mj}(x) &= \int_{S^{d-1}} \phi(\langle \mathbf{x}, \mathbf{y} \rangle)\, h_{mj}(\mathbf{y})\, d\Omega(\mathbf{y}) \\ &\approx \sum_{i \in \mathcal{I}} c_i \phi(\langle \mathbf{x}, \mathbf{y}_i \rangle), \end{aligned}$$

with

$$\sum_{i \in \mathcal{I}} c_i h_{ni}(\mathbf{y}_i) = 0.$$

for any $h_{ni} \in H_{\mathcal{N}}$. Thus, we can approximate a single harmonic by a function of the required form and it is straightforward to generalise this to a linear combination of harmonics.

3. G-INVARIANT POLYNOMIAL DECOMPOSITIONS OF L^2

The general approach to kernels on compact manifolds in Euclidean space which is presented in this section mimics the above proof of Proposition 2.1 in many respects.

All the natural function spaces on the homogeneous space M, such as $\mathbf{C}(M)$ and $L^2(M)$, admit a norm-preserving action of each $g \in G$, often referred to as translation by g, given by $g \cdot f(\mathbf{x}) = f(g^{-1} \cdot \mathbf{x})$. The fact that this is a norm-preserving action on $L^2(M)$ follows from the fact that since G consists of orthogonal matrices, Lebesgue measure and μ are invariant under the action of any $g \in G$. Our main results characterizing the density of $\mathcal{K}$ will follow easily from the results we are about to give providing orthogonal decompositions of $L^2(M)$ into G-invariant spaces of polynomials and relating G-invariant spaces to T_k invariant spaces.

We begin our discussion of decompositions of $L^2(M)$ with a definition of the polynomials P_n which makes no reference to coordinates:

DEFINITION 3.1. The *polynomials of degree at most n* on M are

$$P_n = \text{Span}\left\{\prod_{i=1}^{m} \langle \mathbf{y_i}, \cdot \rangle : \mathbf{y_i} \in \mathbb{R}^d, m \leqslant n\right\} \subset \mathbf{C}(M).$$

The $\mathbf{y_i}$ are not required to be in M but the functions $\langle \mathbf{y_i}, \cdot \rangle$ are considered to be restricted to M.

Now exactly as we did for the sphere we define the harmonic polynomials of degree n:

DEFINITION 3.2. The *harmonics of degree n* are given by $H_n = P_n \cap P_{n-1}^{\perp}$.

Since a linear function $\langle \mathbf{y_i}, \cdot \rangle$ is transformed by the action of $g \in G$ into another linear function, $\langle g \cdot \mathbf{y_i}, \cdot \rangle$, the following invariance and orthogonal decomposition results are immediate, given the density of the polynomials in $\mathbf{C}(M)$.

PROPOSITION 3.3. *The following hold on any embedded homogeneous space M.*

(i) *The spaces P_n and H_n are G-invariant.*

(ii) *The span of the harmonics is dense in* $\mathbf{C}(M)$.

(iii) $L^2(M) = \bigoplus_{n=0}^{\infty} H_n$.

Now we need to connect G-invariant subspaces and kernels. First we note that any linear operator T maps one G-invariant subspace to another G-invariant space exactly when it is G-equivariant according to

DEFINITION 3.4. An operator T on functions is *G-equivariant* provided it commutes with translation by each $g \in G$, i.e. $T(g \cdot f) = g \cdot T(f)$.

Notice that an operator T_k, determined by a kernel k, is G-equivariant exactly when k is a G-invariant kernel, due to the G-invariance of μ. Our hope is to show that each of the H_n is actually T_k invariant. A critical step is the study of a *reproducing kernel* for a subspace $\mathscr{V}$:

DEFINITION 3.5. A kernel $k(\mathbf{x}, \mathbf{y})$ is a *reproducing kernel* for a (closed) subspace $\mathscr{V}$ provided

$$T_k(f) = \begin{cases} f, & \text{for all } \; f \in \mathscr{V} \\ 0, & \text{for all } \; f \in \mathscr{V}^{\perp}. \end{cases}$$

For finite-dimensional spaces the basic facts about reproducing kernels are as follows.

PROPOSITION 3.6. *Let $\mathcal{V}$ be any finite-dimensional subspace of $L^2(M)$, where M is an embedded homogeneous space with associated group G. Then*

(i) *$\mathcal{V}$ has a unique reproducing kernel $k_{\mathcal{V}}(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\dim(\mathcal{V})} f_i(\mathbf{x}) \overline{f_i(\mathbf{y})}$, where $\{f_i\}$ is any orthonormal basis for $\mathcal{V}$.*

(ii) *$T_{k_{\mathcal{V}}}$ is the orthogonal projection on $\mathcal{V}$.*

(iii) *$k_{\mathcal{V}}$ is G-invariant if and only if $\mathcal{V}$ is G-invariant.*

(iv) *If $\mathcal{V}$ is G-invariant, then*

(a) *for any fixed pole $\mathbf{p} \in M$, $\mathcal{V}$ is spanned by the G-translates of the zonal function $k_{\mathcal{V}, \mathbf{p}}$;*

(b) *$k_{\mathcal{V}}(\mathbf{x}, \mathbf{x}) = \dim(\mathcal{V})$, for all $\mathbf{x} \in M$.*

Proof. The formula for $k_{\mathcal{V}}$ satisfies Definition 3.5 by the definition of an orthonormal basis. Moreover, this definition is exactly the definition of the orthogonal projection on $\mathcal{V}$. The uniqueness of $k_{\mathcal{V}}$ follows from the fact that the orthogonal projection is unique. Thus (i) and (ii) are proved.

For (iii) suppose $k_{\mathcal{V}}$ is G-invariant. Then for $f \in \mathcal{V}$, $T_{k_{\mathcal{V}}}(g \cdot f) = g \cdot T_{k_{\mathcal{V}}}(f) = g \cdot f$, since $T_{k_{\mathcal{V}}}$ is the orthogonal projection on $\mathcal{V}$. Hence $g \cdot f \in \mathcal{V}$, i.e., $\mathcal{V}$ is G-invariant. Conversely, if $\mathcal{V}$ is G-invariant, then $\{g \cdot f_i\}$ is also an orthonormal basis and the uniqueness of $k_{\mathcal{V}}$ shows

$$k_{\mathcal{V}}(g^{-1} \cdot \mathbf{x}, g^{-1} \cdot \mathbf{y}) = \sum_{i=1}^{\dim(\mathcal{V})} g \cdot f_i(\mathbf{x}) \overline{g \cdot f_i(\mathbf{y})} = k_{\mathcal{V}}(\mathbf{x}, \mathbf{y}).$$

Thus $k_{\mathcal{V}}$ is G-invariant.

For (iv.a) suppose some $v \in \mathcal{V}$ is orthogonal to all translates $g \cdot k_{\mathcal{V}, \mathbf{p}}$. Then from the fact that $k_{\mathcal{V}}$ reproduces $\mathcal{V}$ and $k_{\mathcal{V}}(\mathbf{y}, \mathbf{x}) = \overline{k_{\mathcal{V}}(\mathbf{x}, \mathbf{y})}$ we get

$$0 = \int_M v(\mathbf{x}) \overline{k_{\mathcal{V}}(g^{-1} \cdot \mathbf{x}, \mathbf{p})} \, d\mu(\mathbf{x})$$

$$= \int_M k_{\mathcal{V}}(g \cdot \mathbf{p}, \mathbf{x}) \, v(\mathbf{x}) \, d\mu(\mathbf{x}) = v(g \cdot \mathbf{p}).$$

Since this holds for all g, $v = 0$ and the translates must span the finite-dimensional space $\mathcal{V}$.

For (iv.b) just note that the G-invariance of $k_{\mathcal{V}}$ means that this kernel is constant on the diagonal. Integration with respect to $\mathbf{x}$ yields $\dim(\mathcal{V})$, since we are assuming μ has been normalized to yield $\mu(M) = 1$. ∎

In the sphere case we saw (2) that the space H_n was invariant under the action of any integral operator with a radial kernel $k(\cdot, \cdot) = \phi(\langle \cdot, \cdot \rangle)$. In the more general context of embedded homogeneous spaces H_n will be invariant under T_k, for k G-invariant, exactly when T_k commutes with the orthogonal projection onto H_n; i.e., if k_n is the G-invariant reproducing kernel for H_n, then H_n is T_k invariant exactly when $T_k T_{k_n} = T_{k_n} T_k$. Thus the significance of our notion of *reflexive spaces* should be clear from the following general commutativity result.

PROPOSITION 3.7. *If M is reflexive then $\mathcal{T}_G$ is a commuting family of operators closed under taking adjoints.*

Proof. We have already noted how the g in Definition 1.1(ii) shows $k(\mathbf{x}, \mathbf{y}) = k(\mathbf{y}, \mathbf{x})$. When this is used in the inner product it shows $\langle T_k(f), h \rangle = \langle f, T_{\bar{k}}(h) \rangle$ for all $f, h \in L^2(M)$, i.e., $T_k^* = T_{\bar{k}}$, so the family is closed under adjoints. Now for any two continuous kernels k_1, k_2, Fubini's theorem shows that

$$T_{k_1} T_{k_2}(f)(\mathbf{x}) = \int_M k_1(\mathbf{x}, \mathbf{z}) \left(\int_M k_2(\mathbf{z}, \mathbf{y}) \, f(\mathbf{y}) \, d\mu(\mathbf{y}) \right) d\mu(\mathbf{z})$$

$$= \int_M \left(\int_M k_1(\mathbf{x}, \mathbf{z}) \, k_2(\mathbf{z}, \mathbf{y}) \, d\mu(\mathbf{z}) \right) f(\mathbf{y}) \, d\mu(\mathbf{y}).$$

So the product $T_{k_1} T_{k_2}$ of two kernel operators is a kernel operator $T_{k_1 * k_2}$ with kernel

$$k_1 * k_2(\mathbf{x}, \mathbf{y}) = \int_M k_1(\mathbf{x}, \mathbf{z}) \, k_2(\mathbf{z}, \mathbf{y}) \, d\mu(\mathbf{z}).$$

Moreover if k_1 and k_2 are G-invariant, then for each pair $\mathbf{x}, \mathbf{y}$ the symmetry of k_1, k_2 and the G-invariance of μ yields

$$k_1 * k_2(\mathbf{x}, \mathbf{y}) = \int_M k_2(\mathbf{y}, \mathbf{z}) \, k_1(\mathbf{z}, \mathbf{x}) \, d\mu(\mathbf{z})$$

$$= \int_M k_2(\mathbf{y}, g^{-1} \cdot \mathbf{z}) \, k_1(g^{-1} \cdot \mathbf{z}, \mathbf{x}) \, d\mu(\mathbf{z})$$

$$= \int_M k_2(g \cdot \mathbf{y}, \mathbf{z}) \, k_1(\mathbf{z}, g \cdot \mathbf{x}) \, d\mu(\mathbf{z})$$

$$= k_2 * k_1(\mathbf{x}, \mathbf{y}),$$

when $g \cdot \mathbf{x} = \mathbf{y}$ and $g \cdot \mathbf{y} = \mathbf{x}$. Thus $T_{k_1} T_{k_2} = T_{k_2 * k_1} = T_{k_2} T_{k_1}$. ∎

We are now in a position to prove our first main result relating polynomials and the analysis of G-invariant kernels.

THEOREM 3.8. *Let M be an embedded reflexive space or more generally an embedded homogeneous space for which the set $\mathcal{T}_G$ is a commuting family of operators. Let $k(\mathbf{x}, \mathbf{y})$ be any continuous G-invariant kernel. Then*

(i) *P_n and H_n are T_k-invariant.*

(ii) *For any choice of pole $\mathbf{p} \in M$ the zonal polynomials in P_n and H_n are also T_k invariant.*

(iii) *Any finite-dimensional T_k invariant subspace $\mathcal{V}$ of $L^2(M)$ consists of polynomials.*

Proof. Since P_n and H_n are G-invariant spaces of continuous functions, (i) follows easily from Propositions 3.7 and 3.6. In fact, T_k commutes with the projection operators arising from the reproducing kernels of either of these two G-invariant spaces. Hence the range of the projections onto either of these two spaces in T_k invariant.

It is simple to see that the zonal function subspaces are T_k invariant. In fact, suppose $h \cdot f = f$ for all h with $h \cdot \mathbf{p} = \mathbf{p}$. Then $h \cdot T_k(f) = T_k(h \cdot f) = T_k(f)$ since T_k is G-equivariant. So $T_k(f)$ is zonal.

For (iii) just note that $P_n^{\perp} \cap \mathcal{V}$ is a decreasing sequence of finite-dimensional T_k-invariant subspaces of $\mathcal{V}$ which must eventually be $\{0\}$. Thus for some large n, $(I - T_{k_{P_n}})\, T_{k_{\mathcal{V}}} = 0$, or $T_{k_{\mathcal{V}}} = T_{k_{P_n}} T_{k_{\mathcal{V}}}$. We conclude that $\mathcal{V}$, the range of the orthogonal projector $T_{k_{\mathcal{V}}}$, is included in P_n. ∎

4. REDUCTION OF ZONAL KERNEL OPERATORS

Characterizing the structure of operators like T_k is really quite simple, once we have the invariance of the (harmonic) polynomial spaces H_n, as in Theorem 3.8. In fact, we just exploit the orthogonal decomposition of an arbitrary function relative to the subspaces H_n,

$$f = \sum_{n=0}^{\infty} f_n, \qquad f_n \in H_n,$$

given in Proposition 3.3. An obvious criterion for the density of the range of T_k is

THEOREM 4.1. *Let $k(\mathbf{x}, \mathbf{y})$ be a G-invariant kernel on a space M such that $\mathcal{T}_G$ is commutative, e.g. a reflexive space. The closure of the range of T_k is all of $C(M)$ if and only if* $\ker (T_k \mid H_n) = \{0\}$ *for all n.*

Proof. In order to approximate an arbitrary continuous function we need only be able to approximate all the elements of H_n for each $n \in \mathbb{N}_0$, because the polynomials are dense in the continuous functions. Because, by Theorem 3.8, H_n is T_k-invariant, we can generate the whole of H_n by action under T_k if and only if T_k is invertible as a finite-dimensional linear operator on H_n. Finally, on a finite-dimensional space, a linear operator is invertible if and only if its nullspace is $\{0\}$. ∎

Of course, this is a very general condition that still requires examination of T_k on all of H_n, for each n. If we exploit the G-equivariance of T_k and thus the T_k invariance of the zonal functions in H_n then we can reduce the analysis of T_k to analysing its restriction to the various zonal harmonic spaces via Theorem 3.8.

We need the following refinement of Proposition 3.3(iii), which gives a complete and detailed analog of Fourier series expansions for (zonal) functions on M for which $\mathscr{T}_G$ is commutative.

PROPOSITION 4.2. *Let M be an embedded compact homogeneous space for which $\mathscr{T}_G$ is a commuting family.*

(i) *Each H_n has a (unique) decomposition $H_n = \bigoplus_{i=1}^{d_n} H_{n,i}$, where each $H_{n,i}$ is a G-irreducible space of polynomials.*

(ii) *Each $f \in L^2(M)$ has an L^2-expansion $f = \sum_n \sum_i f_{n,i}$ with $f_{n,i} = T_{k_{H_{n,i}}}(f) \in H_{n,i}$.*

(iii) *The terms in the expansion in* (ii) *are eigenfunctions for each $T_k \in \mathscr{T}_G$. Specifically,*

(a) *Each $f_{n,i}$ satisfies $T_k(f_{n,i}) = a_{n,i} f_{n,i}$ for some eigenvalue $a_{n,i}$ which depends only on k, n and i.*

(b) *The eigenvalue $a_{n,i}$ calculated from $T_k(k_{H_{n,i},\mathbf{p}}) = a_{n,i} k_{H_{n,i},\mathbf{p}}$ is given by*

$$a_{n,i} \dim(H_{n,i}) = \int_M k(\mathbf{p}, \mathbf{y})\, k_{H_{n,i}}(\mathbf{y}, \mathbf{p})\, d\mu(\mathbf{y}).$$

(c) $k(\mathbf{x}, \mathbf{p}) = \sum_{n=0}^{\infty} \sum_{i=0}^{d_n} a_{n,i}\, k_{H_{n,i}}(\mathbf{x}, \mathbf{p})$.

Proof. Suppose H_n is not G-irreducible, i.e., it has some proper G-invariant subspace $\mathscr{V}$. Then $\mathscr{V}^{\perp} \cap H_n$ is a G-invariant orthogonal complement. Since $\dim(\mathscr{V}) < \dim(H_n) < \infty$, repetition of this argument must eventually reach the irreducible decomposition claimed in (i).

Uniqueness follows from the fact that any G-irreducible $\mathscr{V} \subseteq H_n$ must be equal to one of the $H_{n,i}$ just obtained. In fact $\mathscr{V} \cap H_{n,i} = 0$ or $H_{n,i} = \mathscr{V}$ by the irreducibility of $H_{n,i}$ and $\mathscr{V}$ and the G-invariance of their intersection.

The intersection cannot be 0 for all i, since then $T_{k_{\mathscr{V}}} = \sum_i T_{k_{H_{n,i}}} T_{k_c V} = 0$ from the decomposition just obtained.

Now (ii) refines Proposition 3.3(iii) and follows immediately from the representation of the reproducing kernel for $H_{n,i}$ given by Proposition 3.6.

As for (iii)(a), just note that the G-equivariant operator T_k maps $H_{n,i}$ into itself since $T_k T_{k_{H_{n,i}}} = T_{k_{H_{n,i}}} T_k$ and $T_{k_{H_{n,i}}}$ is the orthogonal projection on $H_{n,i}$. Thus the symmetric T_k has some eigenvector $h \in H_{n,i}$ with real eigenvalue $a_{n,i}$. Now each translate $g \cdot h$ is an eigenvector for the same eigenvalue as $g \cdot T_k(h) = T_k(g \cdot h)$. Since $H_{n,i}$ is G-irreducible, $\operatorname{Span}\{g \cdot h\} = H_{n,i}$. This proves (iii)(a).

For (iii)(b), just observe that for $f(\mathbf{x}) = k_{H_{n,i}}(\mathbf{x}, \mathbf{p})$, $f = f_{n,i} \in H_{n,i}$. So by (iii)(a) this f satisfies the eigenvalue equation $T_k(k_{H_{n,i},\mathbf{p}}) = a_{n,i} k_{H_{n,i},\mathbf{p}}$. Evaluate this at $\mathbf{p}$ and use the fact that $k_{\mathscr{V}}(\mathbf{p}, \mathbf{p}) = \dim(\mathscr{V})$ for any G-invariant $\mathscr{V}$, Proposition 3.6(iv)(b) to get the formula for $a_{n,i}$. Moreover, the sum in (ii) applied to $f = k_{\mathbf{p}}$ yields (iii)(c) since the commutativity of $\mathscr{T}_G$ and (iii)(a) show

$$f_{n,i}(\mathbf{x}) = T_{k_{H_{n,i}}}(k_{\mathbf{p}})(\mathbf{x}) = T_k(k_{H_{n,i},\mathbf{p}})(\mathbf{x}) = a_{n,i} k_{H_{n,i},\mathbf{p}}(\mathbf{x}). \quad \blacksquare$$

Now we combine these last two results to obtain our best criterion for determining the density of the translates of a zonal kernel. This gives a reformulation and proof of Theorem 1.4.

THEOREM 4.3. *Let M be any embedded homogeneous space for which $\mathscr{T}_G$ is commutative, e.g. a reflexive space. Let $k(\mathbf{x}, \mathbf{y})$ be any G-invariant continuous kernel on $M \times M$. Fix a pole $\mathbf{p} \in M$. Then the span of the functions $k(\cdot, \mathbf{y})$, $\mathbf{y} \in M$, is dense in $\mathbf{C}(M)$ if and only if for all n and all zonal polynomials $p \in {}^{K_{\mathbf{p}}}H_n$, $T_k(p) \neq 0$. More specifically, if*

$$k(\mathbf{x}, \mathbf{p}) = \sum_{n=0}^{\infty} \sum_{i=0}^{d_n} a_{n,i} \, k_{H_{n,i}}(\mathbf{x}, \mathbf{p})$$

is the expansion of $k_{\mathbf{p}}$ in terms of the reproducing kernels for the G-irreducible spaces $H_{n,i} \subset H_n$, then the span is dense if and only if $a_{n,i} \neq 0$ for all n, i.

Proof. By Theorem 4.1 we only need to show that $\ker(T_k \mid H_n) = \{0\}$ if and only if $\ker(T_k \mid {}^{K_{\mathbf{p}}}H_n) = \{0\}$. The "only if" direction is trivial. Suppose that $\mathscr{V} = \ker(T_k \mid H_n) \neq \{0\}$. Then $\mathscr{V}$ is G-invariant since T_k is G-equivariant. Hence $\mathscr{V}$ is spanned by the G-translates of the zonal function ${}^{K_{\mathbf{p}}}\mathscr{V}$ and thus, as in Proposition 3.6(iv), $\mathscr{V}$ contains a zonal function $k_{\mathscr{V},\mathbf{p}} \in {}^{K_{\mathbf{p}}}H_n$. Hence $k_{\mathscr{V},\mathbf{p}} \in \ker(T \mid {}^{K_{\mathbf{p}}}H_n) \neq \{0\}$. The more specific result follows from Proposition 4.2 since the $a_{n,i}$ represent all the eigenvalues of T_k restricted to the zonal functions. $\blacksquare$

5. EXAMPLES OF ZONAL KERNEL OPERATORS

We consider the two simple cases of the spheres and the tori.

EXAMPLE 5.1 (The Spheres). Let the point $\mathbf{p}$ be the north pole on the sphere. The sphere is the orbit of $O(d)$ the group of rotations on $\mathbb{R}^d$ and $K_{\mathbf{p}} = O(d-1)$. In this case H_n is the space spanned by the spherical harmonics of degree n. Because k is a zonal kernel $k(\mathbf{x}, \mathbf{y}) = k(g^{-1} \cdot \mathbf{x}, \mathbf{p})$, where $\mathbf{y} = g \cdot \mathbf{p}$. The orthogonal decomposition of $k(\cdot, \mathbf{p})$ with respect to H_n is

$$k(\cdot, \mathbf{p}) = \sum_{n=0}^{\infty} \langle p_n(\cdot, \cdot), k(\cdot, \mathbf{p}) \rangle.$$

However, as $k(\cdot, \mathbf{p})$ is clearly $K_{\mathbf{p}}$ invariant, the above decomposition must also be $K_{\mathbf{p}}$ invariant. It is well known and easy to see that the only elements of H_n which are invariant under rotations fixing the north pole are the multiples of the so-called *zonal harmonic*

$$\begin{aligned} p_n(\mathbf{x}) &= \sum h_{nj}(\mathbf{x})\, h_{nj}(\mathbf{p}) \\ &= p_n^{\lambda}(\langle \mathbf{x}, \mathbf{p} \rangle) \end{aligned}$$

as in (1). Hence

$$\begin{aligned} k(\mathbf{x}, \mathbf{y}) &= (g^{-1} \cdot \mathbf{x}, \mathbf{p}) \\ &= \sum_{n=0}^{\infty} a_n p_n^{\lambda}(\langle g^{-1} \cdot \mathbf{x}, \mathbf{p} \rangle) \\ &= \sum_{n=0}^{\infty} a_n p_n^{\lambda}(\langle \mathbf{x}, \mathbf{y} \rangle), \end{aligned}$$

where

$$a_n = \int_{S^{d-1}} k(\mathbf{x}, \mathbf{p})\, p_n^{\lambda}(\langle \mathbf{x}, \mathbf{p} \rangle)\, d\Omega(\mathbf{x}).$$

We see from the last equation that the only zonal kernels on the sphere are radial kernels (some function of geodesic distance from a fixed point). Furthermore, we can apply exactly the same reasoning as in Proposition 2.1 to see that T_k is invertible on H_n if and only if $a_n \neq 0$.

EXAMPLE 5.2 (The Tori). The torus is not reflexive under the group action of the torus $(\mathrm{SO}(2))^d$, but $\mathscr{T}_G$ is a commuting family of operators.

If we consider the action of $(\mathrm{O}(2))^d$ (realised as outlined below), then the torus is reflexive. In this example we will act on the torus by these two different groups to illustrate how increasing the size of the group G acting on the manifold reduces the size of the family of zonal kernels.

The d-torus $\mathbb{T}^d$ may be identified with the Cartesian product of d unit circles in the complex plane:

$$\mathbb{T}^d = \{(z_1, z_2, ..., z_d) \colon z_i \in \mathbb{C}, |z_i| = 1\}.$$

If we let $\mathbf{p} = (1, 1, ..., 1)$ be the identity on the torus then we can identify the torus with the orbit of $\mathbf{p}$ under the action of $\mathrm{SO}(2)^d$ or $\mathrm{O}(2)^d$. In the former case $K_{\mathbf{p}}$ is trivial. Let r be the element of $\mathrm{O}(2)$ which sends z to $\bar{z}$, and $g_\theta \in \mathrm{O}(2)$, $\theta \in [0, 2\pi)$ be such that $g_\theta \cdot z = e^{i\theta} z$. Set $\mathrm{id} = g_0$. Then, for all $g \in \mathrm{O}(2)$, $g = r^j g_\theta$, where $j = 0, 1$, and $\theta \in [0, 2\pi)$. Then, under the action of $(\mathrm{O}(2))^d$, $\mathbb{T}^d$ is reflexive. In this case, $K_{\mathbf{p}} = \{\mathrm{id}, r\}^d$.

The space H_n is spanned by the functions $\mathbf{z}^\alpha = (z_1^{\alpha_1}, z_2^{\alpha_2}, ..., z_d^{\alpha_d})$, with $\alpha \in \mathbb{Z}^d$ with $|\alpha| = n$, where $|\alpha| = |\alpha_1| + \cdots + |\alpha_d|$. Hence H_n is $K_{\mathbf{p}}$ invariant. The reproducing kernel is

$$p_n(\mathbf{x}, \mathbf{y}) = \sum_{|\alpha| = n} \mathbf{x}^\alpha \bar{\mathbf{y}}^\alpha$$

so that we have the orthogonal decomposition for the zonal kernel

$$\begin{aligned} k(\mathbf{x}, \mathbf{y}) &= k(\mathbf{y}^{-1}\mathbf{x}, \mathbf{p}) \\ &= \sum_{n=0}^{\infty} \langle p_n(\mathbf{y}^{-1}\mathbf{x}, \cdot), k(\cdot, \mathbf{p})\rangle \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} (\mathbf{y}^{-1}\mathbf{x})^\alpha \int_{\mathbb{T}^d} \bar{\mathbf{z}}^\alpha k(\mathbf{z}, \mathbf{p})\, d\mathbf{z} \\ &= \sum_{n=0}^{\infty} \sum_{|\alpha|=n} a_\alpha (\mathbf{y}^{-1}\mathbf{x})^\alpha, \end{aligned}$$

with the obvious definition of a_α.

In the case when $G = (\mathrm{SO}(2))^d$, the zonal kernels on the torus are convolution kernels, and it is clear the T_k is non-singular on H_n if and only if $a_\alpha \neq 0$ for each $|\alpha| = n$. This agrees with the result of Levesley and Kushpel [3].

However, if $G = (0(2))^d$, the kernel k must also be invariant under reflection; i.e., $k(r_i \cdot \mathbf{x}, r_i \cdot \mathbf{y}) = k(\mathbf{x}, \mathbf{y})$, where $r_i \cdot (z_1, ..., z_i, ..., z_d) = (z_1, ..., \bar{z}_i, ..., z_d)$. This means that, if $\beta = (\alpha_1, ..., -\alpha_i, ..., \alpha_d)$, $a_\alpha = a_\beta$, and k is even in each variable of $\mathbf{x}$. This is a more restricted family of zonal kernels than in the previous case. As indicated in Theorem 4.3, T_k is non-singular on H_n if and only if $a_\alpha \neq 0$ for all $|\alpha| = n$ and $\alpha_i \geqslant 0$ for every $i = 1, ..., d$.

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